Optimal Swinging Up of the Pendubot

Miguel Gabriel Villarreal-Cervantes[1*], Carlos Alberto Cruz-Villar[2] and Jaime Alvarez-Gallegos[3]

[1]Mechatronic Section, Postgraduate Department, CIDETEC-IPN, Juan de Dios Bátiz s/n, C.P. 07700 D.F., México.
[2]Electrical Engineering Department, CINVESTAV-IPN, Av. IPN 2508, C.P. 07360 D.F., México.
[3]Research and Posgrade Secretariat, IPN, Luis E. Erro s/n, U. P. Adolfo López Mateos, C.P. 07738 D.F., México.
Correo electrónico: mvillarrealc@ipn.mx
http://www.cidetec.ipn.mx

Abstract — **The swinging up of the pendubot from the rest to the upright position is one of the most difficult control problems, since the system is nonlinear, underactuated and has uncontrollable states. In this paper a continuous control law is obtained by solving a dynamic optimization problem which considers two weighted performance indexes, the time and the energy consumption. The resulting optimal control problem is solved via a gradient like algorithm. Simulation and experimental results are presented.**

I. INTRODUCTION

The Pendubot consists of two links, one of these is actuated (link 1) by a DC motor and the other is not actuated (link 2), see Fig. 1.

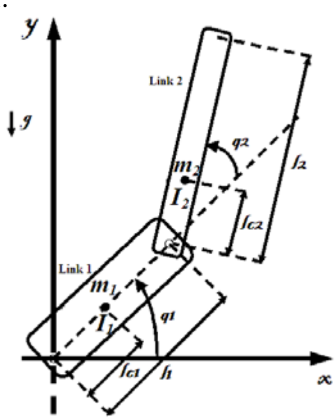

Fig.1 Model of the Pendubot

Considering the state vector as $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$, the swing up control consists on bringing the links from the rest *(-π/2,0,0,0)* to the upright position *(π/2,0,0,0)*.

Several control strategies for swinging up the pendubot have been proposed. Fantoni, Lozano and Spong [6] proposed a control law based on an energy approach and passivity properties. Spong and Block [7], [8] proposed a control law based on partial feedback linearization techniques and Zhao and Yi [4] proposed a bang-bang controller where the control parameters are obtained by a modified genetic algorithm.

In references [6] and [7] minimal time is not a criterion for swinging up the pendubot. In reference [6] only energy consumption is considered as a performance criterion. In reference [4] a bang-bang control law is obtained. The performance criterion only considers the minimal time for the swinging up problem. Due to the dynamic of the actuator is not included in the formulation, the switching from the minimal to the maximal output torque con not be implemented in the experimental results. Hence, simulation results are only carried out.

In this paper a sum weighted performance criterion considering minimal time and minimal energy consumption is proposed to obtain a smooth control law for the experimental results. The optimal control problem is solved by using a gradient like algorithm which satisfies the optimality conditions [1] in this problem.

The rest of the paper is organized as follow: In section II, the dynamic model of the pendubot is presented. Section III states the optimal control problem and presents the gradient like algorithm. In section IV, both simulation and experimental results are presented and discussed. Further work and final conclusions are presented in section V.

II. SYSTEM DYNAMICS

The equations of motion for the Pendubot can be found in [5] or in [8]. In matrix form the equations are:

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) \qquad (1)$$

Where *D(q)* is the symmetric positive definite matrix of inertia, $C(q,\dot{q})\dot{q}$ is the centrifugal and Coriolis matrix and *G(q)* is the matrix of potential energy.

The equation of motion for the pendubot (1) can be formulated as a nonlinear state-space description $\dot{x} = f(x(t), u(t))$ where $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$ is the state vector, q_1 the angle of the link 1 with respect to the horizontal, q_2 the angle of the link 2 with respect to the link 1 and $u(t) = [\tau, 0]^T$ is the input vector.

In Fig. 1 m_1 is the mass of the link 1, m_2 the mass of the link 2, l_1 and l_2 the lengths of link 1 and link 2, l_{c1} the distance to the center of mass of link 1, l_{c2} the distance to the

*Dr. MGVC acknowledges support from COFAA and SIP of the IPN through grant no. 20110165 and SNI-CONACYT.

center of mass of link 2, and I_1 and I_2 the moments of inertia of link 1 and link 2 about their centroids

III. Swing Up Control

Using the Bolza form, the optimization problem is to obtain *u(i)* to minimize the final time and the energy consumption to reach the upright position of the links, i. e. :

$$\underset{u}{Min}\, J = t_f + \int_{t_0}^{t_f} \frac{u^2(t)}{2} dt \quad (2)$$

subject to:

1) Dynamics equation of the Pendubot (3):

$$\dot{x} = f(x,u) \quad (3)$$

2) Initial conditions (rest position) (4):

$$x_1(0) = -\pi/2, \quad x_2(0) = 0, \quad x_3(0) = 0, \quad x_4(0) = 0 \quad (4)$$

3) Terminal constraints ψ (upright position) (5):

$$\begin{aligned} x_1(t_f) - \pi/2 = 0 \quad x_2(t_f) - 0 = 0 \\ x_3(t_f) - 0 = 0 \quad x_4(t_f) - 0 = 0 \end{aligned} \quad (5)$$

If we define the Hamiltonian as in (6) and the auxiliary function in (7), where λ and υ are Lagrange multipliers

$$H(t) = L[u(t)] + \lambda^T(t) f[x(t), u(t)] \quad (6)$$

$$\Phi = [\phi + \upsilon^T \psi]_{t=t_f} \quad (7)$$

where $L[u(t)] = \frac{u^2(t)}{2}$ and $\phi = t_f$.

Following the procedure described in [1], the necessary conditions for a stationary solution is given by the optimality conditions presented in (8)-(9) with the co-state equations (10) and boundary conditions (11).

$$\begin{aligned} 0 &= H_u, \quad t_0 \le t \le t_f \\ 0 &= L_u + \lambda^T f_u \end{aligned} \quad (8)$$

$$\begin{aligned} 0 &= [\dot{\Phi}]_{t=t_f} \\ 0 &= [\Phi_{t_f} + H(t)]_{t=t_f} \end{aligned} \quad (9)$$

$$\dot{\lambda}^T(t) = -H_x(t) = -\lambda^T f_x - L_x \quad (10)$$

$$\lambda^T(t_f) = [\Phi_x]_{t=t_f} = [\phi_x + \upsilon^T \psi_x]_{t=t_f} \quad (11)$$

Numerical Solution with Gradient Method

There are several algorithms to solve the problem described above. These methods are characterized by iterative algorithms for improving estimates of the control histories *u(t)*, so as to come closer to satisfying the optimality conditions and the boundary conditions. In this paper the gradient algorithm is used. It will lead to a local minimum performance index (in the direction of steepest descent).

The development of the gradient algorithm is shown in [1]. This algorithm may be interpreted as finding a small variation in the control functions Δ*u(t)*, and a small change in the final time Δt_f to minimize a linear approximation of Δ*J* subject to a quadratic penalty on Δ*u(t)* and on Δt_f, and to a linear approximation to coming closer to the terminal constraints $\psi=0$, namely:

$$\min_{\Delta u(t), \Delta t_f} \Delta J \approx \dot{\phi} \Delta t_f + \int_0^{t_f} H_u^{\phi} \Delta u dt + \frac{1}{2k}\left[\int_0^{t_f} (\Delta u)^T \Delta u dt + (\Delta t_f)^2\right]$$

subject to

$$\Delta\psi \approx \dot{\psi}\,\Delta t_f + \int_0^{t_f} H_u^{\psi} \Delta u dt = -\eta\psi$$

The steps to use the algorithm are:

1) Choose the step size ($k > 0$ and $0 < \eta \le 1$) and the tolerance or the criterion stop.
2) Guess the final time $t_{f\,initial}$ and the initial control vector $u_{initial}(t)$ for $0 \le t \le t_f$.
3) Forward integration. Compute and store *x(t)* of (3) for $0 \le t \le t_f$.
4) Evaluate ϕ, ψ and determine $\dot{\phi}$, $\dot{\psi}$.
5) Backward integration. Compute and store $\lambda^{\phi}(t)$ and $\lambda^{\psi}(t)$ for $0 \le t \le t_f$:

$$\begin{aligned} \dot{\lambda}^{\phi} &= -f_x^T \lambda^{\phi} - L_x^T \\ \dot{\lambda}^{\psi} &= -f_x^T \lambda^{\psi} \end{aligned} \quad (12)$$

with boundary conditions

$$\begin{aligned} \lambda^{\phi}(t_f) &= [\phi_x^T]_{t=t_f} \\ \lambda^{\psi}(t_f) &= [\psi_x^T]_{t=t_f} \end{aligned} \quad (13)$$

6) Compute $H_u^{\phi}(t)$ and $H_u^{\psi}(t)$ for $0 \le t \le t_f$:

$$\begin{aligned} \left(H_u^{\phi}(t)\right)^T &= f_u^T \lambda^{\phi} + L_u \\ \left(H_u^{\psi}(t)\right)^T &= f_u^T \lambda^{\psi} \end{aligned} \quad (14)$$

7) Determine υ:

$$\upsilon = -Q^{-1} g \quad (15)$$

where

$$\begin{aligned} g &= \dot{\psi}\,\dot{\phi} + \int_0^{t_f} H_u^{\psi} \left(H_u^{\phi}\right)^T dt \\ Q &= \dot{\psi}\,\dot{\psi}^T + \int_0^{t_f} H_u^{\psi} \left(H_u^{\psi}\right)^T dt \end{aligned} \quad (16)$$

8) Compute δ*u(t)* for $0 \le t \le t_f$:

$$\delta u(t) = -k\left[H_u^{\phi} + \upsilon^T H_u^{\psi}\right]^T - \eta\left[H_u^{\psi}\right]^T Q^{-1}\psi \quad (17)$$

9) Compute dt_f:

$$dt_f = -k\left(\dot{\phi} + \upsilon^T \dot{\psi}\right) - \eta \dot{\psi}^T Q^{-1}\psi \qquad (18)$$

10) Compute δu_{avg}:

$$\delta u_{avg} = \sqrt{\frac{1}{t_f}\int_0^{t_f} \delta u^T(t)\delta u(t)dt} \qquad (19)$$

11) If δu_{avg} and $|dt_f|$ and $|\psi|$ are smaller than the tolerance then the minimal control vector function *u(t)* and the minimal final time t_f to swing up the pendubot have been found else, compute the new *u(t)* and t_f, and go to step 3).

$$u(t) = u(t) + \delta u(t)$$
$$t_f = t_f + dt_f \qquad (20)$$

IV. Results

A. *Swinging Up Control*

The solution of the above problem can find several local solution when the initial conditions is changed ($k, \eta, t_{f\ initial}, u_{initial}$). The initial conditions are chosen as: $t_f=2s$, $u_{initial}(t)=1.2Nm$ for $t_0 \le t \le t_f$, $k=0.001$, $\eta=0.001$ and the pendubot parameters of the Pendubot are:

$$m_1 = 0.8436Kg \quad l_{c1} = 0.1573m \quad I_1 = 0.005797Kgm^2$$
$$m_2 = 0.3860Kg \quad l_{c2} = 0.1417m \quad I_2 = 0.005391Kgm^2$$

The sampling time of 10ms for the experimental result is chosen for the swing up control.

In Fig. 2 it is observed that the open loop control swings up the Pendubot from rest position $x=(x_1=-\pi/2, x_2=0, x_3=0, x_4=0)$ to upright position $x=(x_1=\pi/2, x_2=0, x_3=0, x_4=0)$ with a final time t_f around 1.36 seconds furthermore the torque does not exceed of the motor limits (±4Nm).

B. *Balancing Control*

The swinging up control does not maintain the links of the pendubot on the upright position. A balancing control is implemented to keep the links in the upright position. It is designed by linearizing the dynamic equations of the pendubot around the uppermost unstable equilibrium point (upright position) and using a LQR (linear quadratic regulator) in the linear system.

The objective of the LQR [1] is minimizes the quadratic performance index

$$J = \frac{1}{2}\int_0^{\infty}\left(\bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\right)dt$$

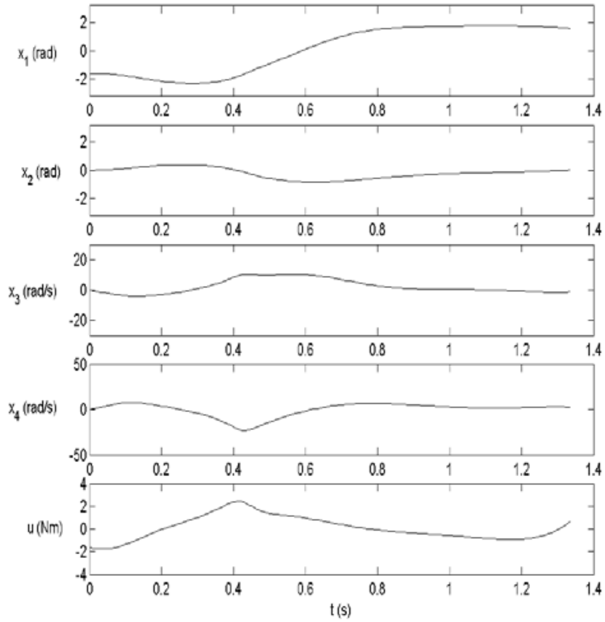

Fig. 2. Simulation of the swinging up control in minimal time and energy consumption.

subject to the linear system around the equilibrium point $x_{reference}=(\pi/2,0,0,0)$

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

where Q and R are the weighting matrices, $\bar{u}$ is the linear control and $\bar{x}$ is the linear state vector ($\bar{x} = x - x_{reference}$).

The design of the LQR is calculated by MatLab utilizing the “lqr” function with the following parameters

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 45.0171 & -15.7595 & 0 & 0 \\ -54.7766 & 74.2633 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 26.1698 \\ -55.5634 \end{bmatrix}$$

$$Q = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad R = 3$$

The lqr function calculates the optimal gain matrix K such that the state feedback law $u=-Kx$ minimizes the quadratic performance index. The gain matrix $K=[-30.5145\ -30.4607\ -7.2518\ -4.7863]$ keeps the links on the upright position.

C. *Combining and Implementing the Controllers*

With the swinging up control and balancing control the control is completed. A switching strategy is used to give the suitable switching to the control signal by watching the states of the system, i.e., when $|x_1 - \pi / 2| < 0.35 rad$ and $|x_2| < 0.17 rad$, the control vector *u(t)* will be the balancing control with a sampling time of 5ms or the other way the control vector will be swing up control with a sampling time of 10ms.

In Fig. 3 the open loop control law (swing up control) takes the first 0.90 seconds to bring the links from rest position to upright position. The control law is switched and the balancing control enters around 0.91 seconds maintaining the links on the upright position.

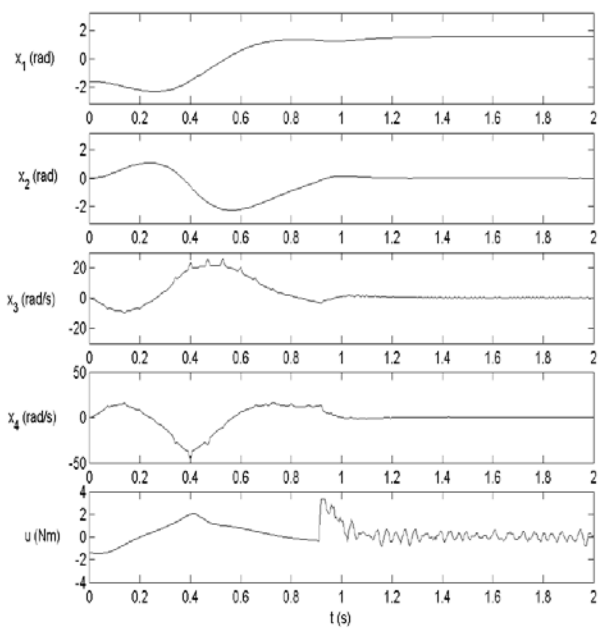

Fig.3. Experimental result of the swing up control and balancing control.

VI. CONCLUSION

We have presented a control strategy for the pendubot that brings the links from the rest position to upright position in minimal time and with minimum energy consumption. The control strategy is based on an optimization theory by proposing a sum weighted performance index which considers minimal time and minimization of energy consumption.

The optimality conditions are satisfied by using a gradient like algorithm which provide a smooth control law.

The swing up control has been tested in simulation and experimental results giving a final time around 1 second.

REFERENCES

[1] A. E. Bryson Jr, *Dynamic Optimization*. Addison Wesley, 1999.
[2] A. E. Bryson Jr and Yu-Chi Ho, *Applied Optimal Control*. Blaisdell Publishing Company, 1969.
[3] D. J. Block, "Mechanical Design and Control of the Pendubot", Master´s thesis, Univ. Illinois, Urbana-Champaign, 1996.
[4] D. ZHAO and J. YI, "Swing up Pendubot with a GA-tuned Bang-Bang Controller", IEEE Int. Conf. on Robotics, Intelligent Systems and Signal Processing, pp. 752-757, Oct. 2003.
[5] M. G. Villarreal, C. A. Cruz and J. Álvarez, *Descripción, modelo, análisis y control del Pendubot,* Boletín UPIITA, No. 28, Septiembre 2011.
[6] I. Fantoni, R. Lozano and M. W. Spong, "Energy based control of the Pendubot", IEEE Transactions on Automatic Control, Vol. 45, No. 4, pp. 725-729, Apr. 2000.
[7] M. W. Spong and D. J. Block, "The Pendubot: A mechatronic system for control research and education", in Proceedings of the 34th IEEE Conf. on Decision and Control, 1995.
[8] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*. John Wiley & Sons. Inc., New York, 1989.